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International Journal of Bifurcation and Chaos, Vol. 28, No. 8 (2018) 1850101 (12 pages)

DOI: 10.1142/S0218127418501018

Loop Numbers for the Stability of Homoclinic Loops of Planar Vector Fields*

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Received ; Revised

This paper is devoted to the study of stability and bifurcations of homoclinic loops for planar vector fields. For a given homoclinic loop, a sequence of loop numbers can be defined such that the stability and bifurcations of the loop are determined by the first nonzero term of the sequence. Formulas for the first several loop numbers were established in the past. In this paper, we will introduce general formulas for the loop numbers for both the single and double homoclinic loops.

Keywords: Homoclinic loops; stability; bifurcation; local moving frame; Poincaré return map.

1. Introduction and Hypotheses

In this paper, we consider a planar C^α vector field

$$\dot{z} = f(z), \quad z = (x, y) \tag{1}$$

or

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{2}$$

where $\alpha \in \mathbb{Z}_+$ is sufficiently large. Suppose that the vector field (1) has an isolated homoclinic loop (or separatrix loop) consisting of a homoclinic orbit

$$\Gamma = \{z = r(t) \,|\, z(\pm\infty) = (0, 0), t \in R\}$$

with a saddle type equilibrium state $O(0, 0)$. Let the corresponding eigenvalues be λ and $-\rho$ with $-\rho < 0 < \lambda$. Our goal is to describe the behavior of trajectories in a sufficiently small neighborhood of Γ.

*This work is supported by NNSFC (No. 11371140) and Shanghai Key Laboratory of PMMP, and supported in part by Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).

1st Reading

$$= \int_{-T}^{T} A_3(t)dt + b_2^2(T)$$

$$= \int_{-T}^{T} A_3(t)dt + B_2^2(T),$$

$$B_4 = b_4(T)$$

$$= \int_{-T}^{T} (A_4 + 3A_3b_2 + 2A_2b_3 + A_2b_2^2)dt$$

$$= \int_{-T}^{T} (A_4 + A_3b_2)dt$$

$$+ \int_{-T}^{T} [2(A_3b_2 + A_2b_3) + A_2b_2^2]dt.$$

Furthermore, the second integral $\int_{-T}^{T}[2(A_3b_2 + A_2b_3) + A_2b_2^2]dt$ can be reduced to

$$2\int_{-T}^{T} (b_2b_3)'dt - 3\int_{-T}^{T} b_2^2b_2'dt = 2B_2B_3 - B_2^3,$$

which yields the expression of B_4. ■

Remark 4.1. When looking for a new loop number, we always assume that all the previous loop numbers are zero. Hence for sequences of $\{B_i\}$ given in Theorem 4, we only need to retain their integral part. In fact, by a recursive process, we can determine the integral expression of B_i one by one. For example, we have

① $B_2 = \int_{-T}^{T} A_2(t)dt,$

② $B_2 = 0, \quad B_3 = \int_{-T}^{T} A_3(t)dt,$

③ $B_2 = B_3 = 0, \quad B_4 = \int_{-T}^{T} (A_4 + A_3b_2)dt$

④ $\vdots$

If we assume that a sequence of saddle quantities is zero, i.e. $c_i = d_i$, then under the transformation $(x, y, t) \to (y, x, -t)$, (4) is invariant under the pre-$2k+1$-order truncation in the neighborhood of $O(0, 0)$. In addition, the solution $u(t)$ of the linear variational equation and the solution $v(t)$ of the adjoint equation are all symmetric at $t = \pm T$, so $(u(t), v(t), z_2(t))$ can be extended symmetrically to the intervals $[T, +\infty)$ and $(-\infty, -T]$. We have the following results

Theorem 5.

$$B_j = \int_{-T}^{T} h_j(t)dt$$

$$= \int_{-\infty}^{+\infty} h_j(t)dt, \quad j = 2, 3, \ldots, \left[\frac{\alpha - 3}{2}\right].$$

Proof. Using the reversible invariance, we obtain

$$\int_{-T-T_0}^{-T} h_j(t)dt + \int_{T}^{T+T_0} h_j(t)dt = 0$$

$$\Rightarrow B_j = \int_{-T}^{T} h_j(t)dt = \int_{-T-T_0}^{T+T_0} h_j(t)d$$

$$= \int_{-\infty}^{+\infty} h_j(t)dt.$$

■

Remark 4.2. For fixed α, we can only obtain $((2p + 3) \le \alpha - 2)$th order normal form in the δ-neighborhood of $O(0, 0)$ (including the Taylor remainder). That is, we can only find the pth order saddle quantity. Therefore, the sequence $\{B_j\}$ only needs to be computed up to B_{p+1}.

5. Poincaré Map and the Criterion of Stability

By (9), and (16)–(18), we have the Poincaré map

$$P = P_1 \circ P_0 : S_0^+ \to S_0^+$$

$$q_0(x_0, \delta) \to q_2 = (x_2, \delta).$$

From (17), we have

$$n_0 = y_1, \quad n(T) = x_2 \tag{20}$$

and (9) leads to the expression of $y_1 = y(x_0)$.

The mapping P uniquely determines the relationship between x_2 and x_0. Note that on S_0^+, $x_0 > 0$. Therefore Γ is unstable if $x_2 - x_0 > 0$, and Γ is stable if $x_2 - x_0 < 0$. Then we only need to consider the sign of the following successor function

$$H(x_0) \triangleq P(x_0) - x_0 = x_2 - x_0.$$

When $R_0, R_1, \ldots, R_{k-1}$ and $B_1, \ldots, B_k$ are all zero, by (9), (18), (20), straightforward calculations can lead to

$$\begin{aligned}H(x_0) &= n(T) - x_0\\ &= y_1 + B_{k+1}y_1^{k+1} + o(y_1^{k+1}) - x_0\\ &= \delta^k k^{-1} x_0^{k+1} D + o(\delta^{2k} x_0^{2k+1} D^2)\\ &\quad + B_{k+1}y_1^{k+1} + o(x_0^{k+1}).\end{aligned}$$

When $R_k \neq 0$, we have

$$\begin{aligned}H(x_0) &= \delta^k k^{-1} x_0^{k+1} k R_k \tau + B_{k+1}x_0^{k+1} + \text{h.o.t.}\\ &= \delta^k R_k x_0^{k+1}\tau + B_{k+1}x_0^{k+1} + \text{h.o.t.}\\ &= (\delta^k R_k \tau + B_{k+1})x_0^{k+1} + \text{h.o.t.}\end{aligned} \tag{21}$$

When $R_k = 0$, $B_{k+1} \neq 0$, we have

$$\begin{aligned}H(x_0) &= \delta^k k^{-1} x_0^{k+1}(\delta k R_{k+1} x_0 \tau)\\ &\quad + B_{k+1}x_0^{k+1} + \text{h.o.t.}\\ &= \delta^{k+1} R_{k+1} x_0^{k+2}\tau + B_{k+1}x_0^{k+1} + \text{h.o.t.}\\ &= (\delta^{k+1} R_{k+1} x_0 \tau + B_{k+1})x_0^{k+1} + \text{h.o.t.}\end{aligned} \tag{22}$$

Since for a fixed $\delta > 0$, $\tau \approx \lambda^{-1} \ln \frac{\delta}{x_0}$ is sufficiently large if x_0 is sufficiently small, it follows from (21) that the sign of $\delta^k R_k \tau + B_{k+1}$ is determined by R_k; while in (22), the sign of $H(x_0)$ is determined by the sign of B_{k+1} because $x_0\tau \approx \delta e^{-\lambda\tau}\tau$ can be sufficiently small. By (22), we know that when the loop numbers $R_0 = B_1 = R_1 = B_2 = R_2 = \cdots = B_k = R_k = 0$, also $B_{k+1} = 0$, the saddle quantity R_{k+1} is just the next quantity to study the stability of homoclinic loop Γ. We obtained the following results.

Theorem 6. *The stability of a homoclinic loop Γ is determined by the first nonzero value in the sequence of loop numbers: $R_0, B_1, R_1, B_2, \ldots, R_k, B_{k+1}, \ldots, R_{l-1}, B_l$. If the first nonzero value is negative (or positive), then the loop is stable (or unstable). Moreover, the length of the sequence depends on α, and is only limited by $l \leq [\frac{\alpha-3}{2}]$.*

We now consider the perturbed system X_μ:

$$\dot{z} = f(z, \mu), \quad z \in R^2, \ \ \mu \in R^\Lambda, \ \Lambda \in N. \tag{23}$$

When the multiparameter $\mu = 0$, system X_μ has an isolated homoclinic loop Γ. Then by [Roussarie, 1986], we have the following lemma.

Lemma 1. *Let $P(x_0)$ be the Poincaré map of x_0. Then for sufficiently small μ, X_μ has an uniform upper bound for the number of limit cycles near Γ, which can be determined as follows. If $P(x_0) - x_0$ is equivalent to $\beta_k x_0^k$ with $\beta_k \neq 0$, then X_μ has at most $2k$ limit cycles for small μ near the homoclinic loop Γ. If $P(x_0) - x_0$ is equivalent to $\alpha_{k+1}x_0^{k+1}\ln x_0$ with $\alpha_{k+1} \neq 0$, then X_μ has at most $2k+1$ limit cycles.*

Note that here Roussarie [1986] considered a C^∞ vector field. But [Leontovich, 1951] and [Joyal & Rousseau, 1989] proved a similar result for C^r-smooth system (r is sufficiently large). Also they gave the estimate for the existence of exact $2k$ (or $2k+1$) limit cycles near Γ.

By (21), we know that when $R_0 = B_1 = R_1 = B_2 = \cdots = B_{k-1} = 0$, $R_k \neq 0$, $P(x_0) - x_0 = \delta^k R_k \tau x_0^{k+1} + \text{h.o.t.} = O(R_k x_0^{k+1} \ln x_0)$ if x_0 is sufficiently small. By (22), when $R_0 = B_1 = R_1 = B_2 = \cdots = B_{k-1} = 0$, and furthermore $R_k = 0$, while $B_{k+1} \neq 0$, we have $P(x_0) - x_0 = B_{k+1}x_0^{k+1} + \text{h.o.t.}$ Then according to bifurcation theory [Chow & Hale, 1982; Han *et al.*, 2018] and Lemma 1, we have the following results.

Theorem 7. *If the first $(k-1)$th loop numbers are zero, but the kth is not, then under small perturbations, at most k limit cycles can be generated near the homoclinic orbit Γ. Moreover, under suitable perturbations, exactly k limit cycles can be generated near Γ.*

6. The Stability of Double Homoclinic Loops

The stability of a double homoclinic loop is related to the stabilities of two single homoclinic loops which share the same saddle point. Assume that the vector field (1) has a double homoclinic loop $\Gamma = \Gamma_1 \cup \Gamma_2$ with a hyperbolic saddle point O. We also assume the loop is isolated, that is, there is no periodic orbit in a neighborhood of Γ. Since we have already discussed the inner stability of single homoclinic loops, in this section we only study the outer stability of double homoclinic loops. Similar to the single homoclinic loop, if there exists an outer neighborhood U of Γ such that $\omega(A) = \Gamma$ (or $\alpha(A) = \Gamma$) for any point A in U, then Γ is said to be outer-stable (or outer-unstable). By using normal form theory and Poincaré maps, we will give a sequence of constants to determine its outer stability.

Following the discussion in Sec. 5, we can define the stability of a double homoclinic loop Γ by introducing the double homoclinic loop numbers. Below is our main result of this section.

Theorem 8. *The stability of a double homoclinic loop is determined by the first nonzero value in the sequence of loop numbers*: $R_0^*, B_1^*, R_1^*, B_2^*, \ldots, R_k^*, B_{k+1}^*, \ldots, R_{l-1}^*, B_l^*$. *If the first nonzero value is negative, then the loop is outer-stable. Otherwise, if it is positive, then the loop is outer-unstable.*

It follows from [Han & Zhu, 1994; Han *et al.*, 2003; Han & Wu, 2004] that

$$R_0^* = R_0 = (P_x + Q_y)|_{(0,0)},$$

$$B_1^* = \sum_{i=1}^{2} \oint_{\Gamma_i} (P_x + Q_y)|_{\Gamma_i} dt, \quad (\text{if } R_0^* = 0),$$

$$R_1^* = -R_1, \quad (\text{if } R_0^* = B_1^* = 0).$$

In this section, we will establish formulas for the entire sequence of the double homoclinic loop numbers, and demonstrate the relation between the stability criteria for the double homoclinic loops and the single homoclinic loops. The proof follows from the arguments developed in the previous sections for single homoclinic loops.

Proof. **Outline of the proof:** To prove Theorem 8, we still work on the normal form (4). Similar to the single homoclinic loop, we can construct the moving coordinate frames along Γ_i, $i = 1, 2$. We can define local Poincaré maps near the saddle point O and the nonlocal Poincaré maps along the homoclinic loop Γ_i in neighborhood of the double homoclinic loop Γ. Then we have the following expressions of the corresponding Poincaré map along Γ_1:

When $R_k \neq 0$, we have

$$P_{11}(x_0) = x_0 + \delta^k R_k x_0^{k+1} \tau_1 + \text{h.o.t.} \tag{24}$$

When $R_k = 0$, $B_{k+1,1} \neq 0$, we have

$$P_{11}(x_0) = x_0 + B_{k+1,1} x_0^{k+1} + \text{h.o.t.} \tag{25}$$

In the same way, we have the following expressions of the corresponding Poincaré map along Γ_2:

When $R_k \neq 0$, we have

$$P_{21}(x_2) = x_2 + \delta^k R_k x_2^{k+1} \tau_2 + \text{h.o.t.} \tag{26}$$

When $R_k = 0$, $B_{k+1,2} \neq 0$, we have

$$P_{21}(x_2) = x_2 + B_{k+1,2} x_2^{k+1} + \text{h.o.t.} \tag{27}$$

Here $B_{k+1,i}$ is the integral form given in Theorem 6 along Γ_i, $i = 1, 2$. A straightforward calculation gives the successor functions along the double homoclinic loops Γ:

$$P(x_0) = P_{21} \circ P_{11}(x_0) - x_0$$

$$= \begin{cases} \delta^k(\tau_1 + \tau_2) R_k x_0^{k+1} + \text{h.o.t.}, \\ \qquad\qquad \text{as } R_k \neq 0, \\ (B_{k+1,1} + B_{k+1,2}) x_0^{k+1} + \text{h.o.t.}, \\ \qquad\qquad \text{as } R_k = 0. \end{cases} \tag{28}$$

Denote $R_k^* = (-1)^k R_k$, $B_{k+1}^* = (-1)^k (B_{k+1,1} + B_{k+1,2})$, $k = 0, 1, 2, \ldots$. Note that $x_0 < 0$ on S_0^-. Then the conclusions of Theorem 8 follow directly. ∎

Remark 6.1. For $k = 1$, $R_1^* = -R_1$, and for $k = 2$, $R_2^* = R_2$. These cases are consistent with the results in [Han *et al.*, 2003; Han & Zhu, 2007].

Acknowledgments

The authors would like to thank the referees for their helpful comments and suggestions.

References

Andronov, A. A., Leontovich, E. A., Gordon, I. E. & Maier, A. G. [1971] "The theory of bifurcations of dynamical systems on a plane," Israel Program of Scientific Translations, Jerusalem.

Chen, S. & Du, D. [2015] "Stability and perturbations of homoclinic loops in a class of piecewise smooth systems," *Int. J. Bifurcation and Chaos* **9**, 1550114.

Chow, S. N. & Hale, J. K. [1982] *Methods of Bifurcation Theory* (Springer, NY).

Dulac, H. [1923] "On limit cycles," *Bull. Soc. Math. France* **51**, 45–188.

Dumortier, F. & Li, C. Z. [2001] "Perturbations from an elliptic Hamiltonian of degree four. I: Saddle loop and two saddle cycle," *J. Diff. Eqs.* **176**, 114–157.

Feng, B. & Qian, M. [1985] "The stability of separatrix loops and limit cycle bifurcations," *Acta Math. Sin.* **28**, 53–70.

Han, M. & Zhu, D. [1994] *Bifurcation Theory of Differential Equations (in Chinese)* (Coal Mine Industry Publishing House, Beijing).

Han, M., Hu, S. & Liu, X. [2003] "On the stability of double homoclinic and heteroclinic cycles," *Nonlin. Anal.* **53**, 701–713.

Han, M. & Wu, Y. [2004] "The stability of double homoclinic loops," *Appl. Math. Lett.* **17**, 1291–1298.

Han, M. & Zhu, H. [2007] "The loop quantities and bifurcations of homoclinic loops," *J. Diff. Eqs.* **234**, 339–359.

Han, M., Sheng, L. & Zhang, X. [2018] "Bifurcation theory for finitely smooth planar autonomous differential systems," *J. Diff. Eqs.* **264**, 3596–3618.

Joyal, P. [1988] "Generalized Hopf bifurcation and its dual: Generalized homoclinic bifurcation," *SIAM J. Appl. Math.* **48**, 481–496.

Joyal, P. & Rousseau, C. [1989] "Saddle quantities and applications," *J. Diff. Eqs.* **78**, 374–399.

Krupa, M. & Melbourne, I. [2004] "Asymptotic stability of heteroclinic cycles in systems with symmetry. II," *Proc. Roy. Soc. Edinburgh.* **134A**, 1177–1197.

Leontovich, E. A. [1951] "On birth of limit cycles from separatrices," *DAN. SSSR* **78**, 641–644.

Li, J. B. [2003] "Hilbert's 16th problem and bifurcations of planar polynomial vector fields," *Int. J. Bifurcation and Chaos* **13**, 47–106.

Liu, X. B. & Zhu, D. M. [2012] "On the stability of homoclinic loops with higher dimensions," *Discr. Contin. Dyn. Syst. B* **17**, 915–932.

Liu, X. B., Wang, Z. Z. & Zhu, D. M. [2012] "Bifurcation of rough heteroclinic loop with orbit flips," *Int. J. Bifurcation and Chaos* **22**, 1250278-1.

Liu, X. B., Liu, J. Y. & Zhu, D. M. [2013] "Degenerate bifurcations of heterodimensional cycles with orbit flip," *Int. J. Bifurcation and Chaos* **23**, 1350080.

Luo, H. Y. & Li, J. B. [2005] "What are the separatrix values named by Leontovich on homoclinic bifurcation," *Appl. Math. Mech. (English Ed.)* **26**, 457–464.

Melnikov, B. K. [1963] "On the stability of the center for periodic perturbation of time," *Trans. Moscow Math. Soc.* **12**, 1–57.

Ragazzo, C. G. [1997] "On the stability of double homoclinic loops," *Commun. Math. Phys.* **184**, 251–272.

Roussarie, R. [1986] "On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields," *Bol. Soc. Bras. Mat.* **17**, 67–101.

Shilnikov, L. P., Shilnikov, A. L., Turaev, D. V. & Chua, L. O. [2001] *Methods of Qualitative Theory in Nonlinear Dynamics, Part II* (World Scientific, Singapore).

Zhao, L. & Wang, X. [2009] "The separatrix values of a planar homoclinic loop," *Int. J. Bifurcation and Chaos* **7**, 2233–2247.

Zhu, D. M. & Xia, Z. [1998] "Bifurcations of heteroclinic loops," *Sci. China Ser. A* **41**, 837–848.

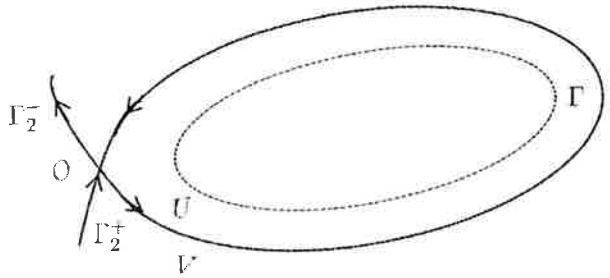

Fig. 1. Homoclinic loop Γ is within an annulus consisting of an outer neighborhood V and an inner neighborhood U.

A small neighborhood of Γ is an annulus, divided by Γ into two regions, U and V, see Fig. 1. In the outer region V, there are two segments of the local stable and unstable manifolds of O which are part of the separatrices Γ_2^+ and Γ_2^-. All the other trajectories in V must leave V as $t \to \pm\infty$. Hence, we will study the stability of Γ from the inner region U only. In this context, we say that the homoclinic loop is asymptotically one-side stable if it is the ω-limit of all the trajectories starting from U. We say that the homoclinic loop is unstable if it is the α-limit of all the trajectories in U.

The study of the stability and bifurcations of homoclinic loops can be traced back to [Dulac, 1923]. Since then great progress has been made. Shilnikov *et al.* [2001] defined a sequence $\sigma_0, s_1, \sigma_1, \ldots, s_n, \sigma_n, s_{n+1}, \ldots$ and proved that the stability of a separatrix loop is determined by the sign of the first nonzero entry in that sequence, if not all entries in the sequence are zero. But they did not give detailed expressions for the entries of the sequence. Similar results were obtained in [Joyal, 1988]. Concrete expressions for the first several entries of the sequence were obtained by some authors. For example, see the works of [Andronov *et al.*, 1971; Chow & Hale, 1982; Feng & Qian, 1985; Han & Zhu, 2007; Leontovich, 1951; Melnikov, 1963; Zhao & Wang, 2009]. For more references about the stability of homoclinic loop, see [Chen & Du, 2015; Dumortier & Li, 2001; Han *et al.*, 2003; Han & Wu, 2004; Liu & Zhu, 2012; Krupa & Melbourne, 2004; Ragazzo, 1997; Roussarie, 1986] and the references cited therein. In this paper, we will show that for a homoclinic loop of planar vector fields, the first nonzero entry in the sequence may not always be determined by the vector field near the saddle point, but may also depend on the vector field near the homoclinic loop. Hence the entries of the sequence will be called the homoclinic loop numbers.

In addition to being a useful tool to study the bifurcations of homoclinic orbits, the homoclinic loop numbers can also be used to study Hilbert's 16th problem (see [Li, 2003]). As pointed out in [Joyal, 1988], the homoclinic loop numbers can be used to determine the number of limit cycles that can bifurcate from a homoclinic loop. See Theorem 7 in Sec. 5 of this paper. Therefore it is important to find concrete expressions of the homoclinic loop numbers, both for theoretical studies, and for practical applications.

The stability of a planar separatrix loop is easily solved when the divergence at the saddle point

$$\sigma_0 = (P_x + Q_y)|_{(0,0)}$$

is nonzero. In this case, Dulac in 1923 studied analytic systems and obtained the following result.

Theorem 1. *If $\sigma_0 < 0$, the homoclinic loop Γ is asymptotically one-side stable. If $\sigma_0 > 0$, the homoclinic loop Γ is unstable.*

The same results were probed by [Chow & Hale, 1982; Shilnikov *et al.*, 2001] when the systems are C^r-smooth with $r \geq 2$. If the divergence $\sigma_0 = \lambda - \rho = 0$, then the equilibrium point $O(0,0)$ is called a weak saddle. In this case, it is impossible to determine the stability of the homoclinic cycle by the vector field near the saddle point alone. It follows from [Feng & Qian, 1985; Leontovich, 1951] that we have

Theorem 2. *If $\sigma_0 = 0$ and $s_1 = \int_{-\infty}^{+\infty}(P_x + Q_y)|_\Gamma dt < 0$ (> 0), then the homoclinic loop is stable (resp., unstable).*

To discuss the case when $\sigma_0 = s_1 = 0$ and also for the purpose of presenting our main results, we need to reduce system (2) near the weak saddle to a more refined normal form of the system. By [Han & Zhu, 2007; Joyal, 1988], system (2) can then be written as

$$\begin{aligned} \dot{x} &= x\left[\lambda + \sum_{i=1}^{\ell} c_i(xy)^i\right] + o(|x^2+y^2|)^{\frac{r}{2}}, \\ \dot{y} &= -y\left[\lambda + \sum_{i=1}^{\ell} d_i(xy)^i\right] + o(|x^2+y^2|)^{\frac{r}{2}}, \end{aligned} \tag{3}$$

where $\ell = [\frac{r-1}{2}]$, $r \leq \alpha - 1$. Denote $R_i = c_i - d_i$ $(i > 0)$, then R_i is called the ith saddle quantity of (2) at $O(0, 0)$, by [Joyal & Rousseau, 1989], they also provided formulas for saddle quantities.

When $\sigma_0 = s_1 = 0$, it is shown in [Han & Zhu, 1994] that the third entry in the sequence is the first order saddle quantity R_1 which determines the stability of Γ(see also [Shilnikov *et al.*, 2001]).

To summarize, for a given homoclinic loop of a planar vector field, the first loop number is the divergence of the vector field at the saddle point [Chow & Hale, 1982; Dulac, 1923; Shilnikov *et al.*, 2001], the second loop number (assuming the first one is zero) is the integral of the divergence around the loop [Andronov *et al.*, 1971; Feng & Qian, 1985]. If the first two loop numbers vanish, the third number is the first order saddle quantity R_1 [Han & Zhu, 1994].

To generalize the previously obtained results, in this paper we discuss the stability of a given homoclinic loop of a planar vector field by a sequence of numbers

$$R_0, B_1, R_1, B_2, R_2, B_3, R_3, \ldots,$$
$$R_k, B_{k+1}, \ldots, R_{l-1}, B_l, R_l,$$

where B_j is the $2j$th loop number, and the jth saddle quantity R_j is the $2j + 1$th loop number (see [Shilnikov *et al.*, 2001]), $j = 1, 2, \ldots, l$, (in particular, $R_0 = \sigma_0$, $B_1 = s_1$). The expressions of $B_2, B_3, \ldots, B_l$ will be given below. The saddle quantities R_j are related to the normal form near the saddle, and the separatrix quantities B_j are related to the nonlocal map near the homoclinic loop. The alternation of these quantities determines the stability of the homoclinic loop. Note that the formulas of the saddle quantities have already been obtained in [Joyal & Rousseau, 1989]. In the present work, we focus on computing the separatrix quantities and demonstrating how to use the saddle quantities and the separatrix quantities to discuss the stability of homoclinic loops.

The computation of separatrix quantities B_j in planar systems is a difficult problem. In principle, these formulas are integrals along the homoclinic loop which is similar to the Melnikov integral, but they are more difficult to compute in practice. Great progress has been made due to many new ideas in dynamical systems. Using the traditional curvilinear coordinate system, Han and Zhu [2007] established the formula for the second separatrix quantity when the first three homoclinic loop numbers are zeros. The third separatrix quantity was obtained in [Luo & Li, 2005] by Tkachev's method. Recently the third and fourth separatrix quantities were obtained in [Zhao & Wang, 2009] using the traditional curvilinear coordinate system.

In this paper, we will obtain the expressions of the other separatrix quantities by constructing a moving coordinate frame in the neighborhood of the homoclinic loop Γ, and furthermore obtain complete results on detailed expressions of the whole homoclinic loop numbers. The moving coordinate frame was originally introduced in the homoclinic and heteroclinic bifurcation problems in higher dimensional spaces (see [Liu *et al.*, 2012; Liu *et al.*, 2013; Zhu & Xia, 1998]), and was modified to discuss the stability of homoclinic loops. Compared with the traditional curvilinear coordinate system, the moving coordinate frame is taken from the tangent vector bundle along the homoclinic orbit and the solution vector bundle of the linearized system, and we do not unitize or regularize the coordinate system. Therefore, the new coordinate system can fully reflect the geometric properties of the dynamical system, such as the invariance, exponential compressibility and expansibility of stable and unstable manifolds, etc.

This paper is organized as the follows. In Sec. 2, we obtain the local Poincaré map near the origin by using the normal form for the weak saddle. In Sec. 3, we establish the moving coordinate frame along the homoclinic loop by the solutions of linear variational equations. In Sec. 4, we obtain the nonlocal Poincaré map and derive the expressions of the separatrix quantities. In Sec. 5, we prove the main results and show that the homoclinic loop numbers are divided into two classes: saddle quantities and separatrix quantities. The alternation of these quantities determines the stability of the homoclinic loop. The homoclinic loop numbers for double homoclinic loops will be presented in the last section.

2. Local Poincaré Map Near the Origin

A basic method for studying the stability of homoclinic loop is to construct a Poincaré map by the combination of two maps: a local map near the saddle and a nonlocal map along the homoclinic loop outside a neighborhood of the saddle. As for the local map, the trajectories come arbitrarily close to the saddle. Therefore, the flight time is not bounded from above. Then we need to give an appropriate

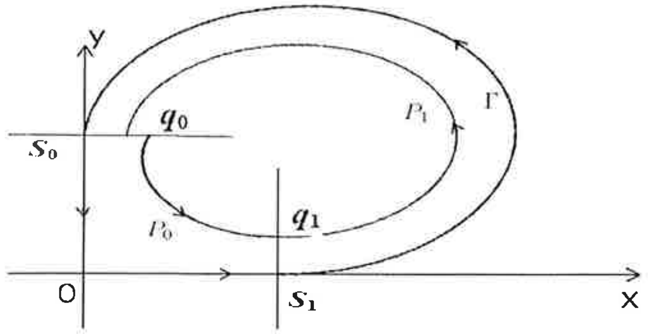

Fig. 2. The Poincaré map near the homoclinic loop Γ.

estimate for the local map. To do this we will reduce the system in a neighborhood of O to a special form.

Consider (1) and (2). Under the assumptions of the previous section, the local unstable manifolds and the local stable manifolds of $O(0,0)$ restricted to Γ are tangent to the x and y axes, respectively, and can be expressed as:

$$W^u_{\text{loc}} = \{z = (x, y) \,|\, y = y^u(x), 0 \le x \le 2\delta\},$$
$$W^s_{\text{loc}} = \{z = (x, y) \,|\, x = x^s(y), 0 \le y \le 2\delta\},$$

where y^u, x^s are $C^{\alpha-1}$-smooth functions. Assume that (1) has a homoclinic loop with $-\rho + \lambda = 0$, i.e. the saddle is weak. It is known that system (1) can be transformed into the following normal form (see [Han & Zhu, 2007; Joyal, 1988])

$$\begin{aligned} \dot{x} &= x\left[\lambda + \sum_{i=1}^{\ell} c_i(xy)^i + (xy)^\ell w_1(x,y)\right], \\ \dot{y} &= y\left[-\lambda - \sum_{i=1}^{\ell} d_i(xy)^i + (xy)^\ell w_2(x,y)\right], \end{aligned} \tag{4}$$

where $|(x,y)| < 2\delta$, $|w_i| = o(1)$, $w_i \in C^{\alpha-2\ell-3}$, $i = 1, 2$, $0 < \delta \ll 1$.

Along the homoclinic loop $z = r(t)$, for a small $\delta > 0$, define two cross-sections S_0, S_1 as follows.

$$S_0 = \{z = (x,y) \,|\, y = \delta, 0 \le |x| < \varepsilon \ll \delta^2\},$$
$$S_1 = \{z = (x,y) \,|\, x = \delta, 0 \le |y| < \varepsilon \ll \delta^2\}.$$

Assume that the flight time from S_1 to S_0 is $2T$ where $r(-T) \in S_1$ and $r(T) \in S_0$. It is easy to see that T can be arbitrarily large if $\delta > 0$ is sufficiently small. For such small δ and large T, from the definitions of S_0, S_1, we have $r(-T) = (\delta, 0)$, $r(T) = (0, \delta)$.

We will study the flow in the inner region U only. Let the restrictions of S_0, S_1 to U be

$$S_0^+ = \{z \in S_0 \,|\, x \ge 0\},$$
$$S_1^+ = \{z \in S_1 \,|\, y \ge 0\}.$$

The solution of system (4) induces the local Poincaré map

$$P_0 : q_0 \in S_0^+ \to q_1 \in S_1^+,$$
$$q_0 = (x_0, \delta), \quad q_1 = (\delta, y_1).$$

Let $\tau = \tau(x_0)$ be the time spent from q_0 to q_1. It is obvious that $\tau(x_0)$ is a strictly monotone function and $x_0 \to 0$ if and only if $\tau \to +\infty$. If we define $y_1 = 0$ when $x_0 = 0$. Then the function $y_1 = y(x_0)$ induced by the map P_0 is continuous at $x_0 = 0$ (see [Shilnikov *et al.*, 2001]). Suppose the order of the weak saddle $O(0,0)$ is not less than ℓ, $\alpha \ge 2\ell + 5$, then for any $k < \ell$, set $R_k = c_k - d_k$. Suppose R_k is the first nonzero loop number, that is,

$$R_0 = B_1 = R_1 = B_2 = R_2 = \cdots = B_k = 0, \quad R_k \ne 0.$$

In this case, the saddle quantity is the first nonzero term in the sequence of loop numbers. Multiply the first equation of (4) by y and the second equation by x, then add them together, we have

$$\begin{aligned} \frac{d(xy)}{(xy)^{k+1}} &= [R_k + o(1)]dt, \quad \text{if } R_k \ne 0, \\ \frac{d(xy)}{(xy)^{k+1}} &= [R_{k+1} + o(1)]xydt, \\ &\quad \text{if } R_k = 0, \ R_{k+1} \ne 0. \end{aligned} \tag{5}$$

When $R_k \ne 0$, from the first equation, we can determine the stability of homoclinic loop Γ by the sign of the saddle quantity R_k. But when $R_k = 0$, $R_{k+1} \ne 0$, combining with the second equation, we will show that there may exist a separatrix quantity B_{k+1} between the saddle quantities R_k, R_{k+1}. If $B_{k+1} \ne 0$, the sign of B_{k+1} can determine the stability of homoclinic loop Γ. Also we know that when the separatrix quantity $B_{k+1} = 0$, the saddle quantity R_{k+1} is just the next number to determine the stability of homoclinic loop Γ, if $R_{k+1} \ne 0$. See Sec. 5 for details.

Integrate the equations along the trajectory from q_0 to q_1, and apply the integral mean value theorem to the right-hand side of the above equation, we have

$$[x_0y_0]^{-k} - [x_1y_1]^{-k} = D_k \triangleq k[R_k + o(1)]\tau, \quad \text{if } R_k \neq 0,$$

$$[x_0y_0]^{-k} - [x_1y_1]^{-k} = D_k^* \triangleq k[R_{k+1} + o(1)]x(t^*)y(t^*)\tau, \quad \text{if } R_k = 0, \ R_{k+1} \neq 0, \tag{6}$$

where $y_0 = x_1 = \delta$, $t^* \in (0,\tau)$, $R_0 = B_1 = R_1 = \cdots = B_k = 0$. Dividing both sides of the second equation of (5) by xy and solving it directly from $t = 0$ to $t = t^*$ and taking a Taylor expansion, we have

$$x(t^*)y(t^*) = x_0y_0[1 - (k+1)R_{k+1}(x_0y_0)^{k+1}t^*]^{-\frac{1}{k+1}} = x_0y_0 + o((x_0y_0)^{k+2}t^*).$$

Substituting the expression of $x(t^*)y(t^*)$ into D_k^*, we obtain

$$\begin{aligned} D_k^* &= k[R_{k+1} + o(1)]x_0y_0[1 + o(x_0y_0)^{k+1}t^*]\tau \\ &= k\delta[R_{k+1} + o(1)]x_0[1 + o(x_0)t^*]\tau \\ &= k\delta[R_{k+1} + o(1) + o(x_0)t^*]x_0\tau. \end{aligned} \tag{7}$$

Taking $D = D_k$ if $R_k \neq 0$, and $D = D_k^*$ if $R_k = 0$, $R_{k+1} \neq 0$, then by (6), we obtain the unified expression

$$y_1 = x_0[1 - x_0^k\delta^k D]^{-\frac{1}{k}}. \tag{8}$$

Since $x_0 \approx \delta e^{-\lambda\tau}$, and $x_0 \to 0$ if and only if $\tau \to +\infty$, then $x_0\tau t^* \approx \delta e^{-\lambda\tau}\tau t^* \to 0$, as $x_0 \to 0$, furthermore, we have $x_0^k\delta^k D \to 0$ as $x_0 \to 0$. With the Taylor expansion, (8) can then be written as

$$y_1 = x_0 + \delta^k k^{-1} x_0^{k+1} D + o(\delta^{2k}x_0^{2k+1}D^2), \tag{9}$$

where

$$D = \begin{cases} D_k = k[R_k + o(1)]\tau, & \text{if } R_k \neq 0, \\ D_k^* = \delta k[R_{k+1} + o(1) + o(x_0)t^*]x_0\tau, & \text{if } R_k = 0, \ R_{k+1} \neq 0. \end{cases}$$

Then the local Poincaré map $P_0 : S_0^+ \to S_1^+$ is uniquely determined by (9).

Remark 2.1. In (9), we give the expression in different precisions according to whether $R_k = 0$ or not. The difference in accuracy is mainly due to our selection of different degrees of truncation of the normal form (4). This technique and the resulting expression of P_0 can make the complex process of studying the stability of Γ much simpler.

3. Solutions of the Linear Variational Equation

Next we establish the moving frame along the homoclinic loop Γ outside a neighborhood of the origin. We will consider the linear variational equation of (1) along Γ, and obtain some solutions with particular property in the moving coordinate frame. Consider the following linear variational equation

$$\dot{z} = A(t)z, \quad A = Df(r(t)), \quad t \in [-T, T] \tag{10}$$

and its adjoint system

$$\dot{z} = -A^*(t)z, \quad t \in [-T, T], \tag{11}$$

where $*$ denotes the transpose of a matrix. By the discussion in Sec. 2, we know that $r(-T) = (\delta, 0)^*$, $r(T) = (0, \delta)^*$. Combining with (4), we have

$$\dot{r}(-T) = (\lambda\delta, 0)^*, \quad \dot{r}(T) = (0, -\lambda\delta)^*.$$

Then we can choose a fundamental solution matrix $Z(t) = (u(t), z_2(t))$ of (10), such that

$$z_2(t) = -\frac{\dot{r}(t)}{\lambda\delta}, \quad Z(T) = I_2,$$

$$Z(-T) = \begin{pmatrix} \omega_{11} & -1 \\ \omega_{12} & 0 \end{pmatrix},$$

where I_2 is second order unit matrix. Denote $B_1 = \int_{-\infty}^{+\infty}(P_x + Q_y)|_{z=r(t)}dt$, we have

Theorem 3. *$\omega_{12} > 0$, $(1 - \omega_{12})B_1 > 0$, $\omega_{12} = 1$ iff $B_1 = 0$, and $\lim_{\delta\to 0}\frac{\omega_{11}}{\omega_{12}} = 0$.*

Proof. Let $u(t)$ be the solution of (10) with the initial value $(1, 0)$ at $t = T$. Then by the existence and uniqueness theorem, there exists unique $(\omega_{11}, \omega_{12})^*$, such that $u(-T) = (\omega_{11}, \omega_{12})^*$. By virtue of Liouville formula about Wronski determinant, we have

$$\begin{aligned} \omega_{12} &= \det Z(-T) \\ &= \exp\left\{-\int_{-T}^{T} \operatorname{div} Df(z)\Big|_{z=r(t)} dt\right\} \det Z(T) \\ &= \exp\left\{-\int_{-T}^{T} \operatorname{div} Df(z)\Big|_{z=r(t)} dt\right\} \\ &= \exp\left\{-\int_{-\infty}^{+\infty} (P_x + Q_y)\Big|_{z=r(t)} dt\right\} \\ &= \exp\{-B_1\}, \end{aligned}$$

then we obtain $\omega_{12} > 0$, $(1 - \omega_{12})B_1 > 0$, and $\omega_{12} = 1$ iff $B_1 = 0$.

Before proving the limit, we first consider the extension of the solutions $u(t)$ and $z_2(t)$ to the intervals $[T, T + T_1]$ and $[-T - T_1, -T]$ respectively. Note that the unstable and stable manifolds have been straightened to x and y axes, respectively. Observe that the trajectories of (4) have symmetry when restricted to the 2δ-neighborhood of x and y axes near the origin. Thus there exists T_1 and δ_1, such that

$$r(T+T_1) = (0, \delta_1)^*, \quad r(-T-T_1) = (\delta_1, 0)^*$$

and $\delta_1 \to 0$ if and only if $T_1 \to +\infty$. Let $r(t) = (r_x(t), r_y(t))^*$, $u(t) = (x(t), y(t))^*$, then we have

$$A(t) = \begin{pmatrix} \lambda & 0 \\ d_1 r_y^2(t) & -\lambda \end{pmatrix}, \quad r(t) = (0, r_y(t))^*,$$

$$r_y(t) = \delta e^{-\lambda(t-T)}, \quad t \in [T, T+T_1],$$

$$A(t) = \begin{pmatrix} \lambda & c_1 r_x^2(t) \\ 0 & -\lambda \end{pmatrix}, \quad r(t) = (r_x(t), 0)^*,$$

$$r_x(t) = \delta e^{\lambda(t+T)}, \quad t \in [-T-T_1, -T],$$

where $\delta_1 = \delta e^{-\lambda T_1}$. Written in component form, (10) can be expressed as

$$\begin{cases} \dot{x} = \lambda x, \\ \dot{y} = d_1 \delta^2 e^{-2\lambda(t-T)} x - \lambda y, \end{cases}$$

$$u(T) = (1, 0)^*, \quad t \in [T, T+T_1]$$

and

$$\begin{cases} \dot{x} = \lambda x + c_1 \delta^2 e^{2\lambda(t+T)} y, \\ \dot{y} = -\lambda y, \end{cases}$$

$$u(-T) = (\omega_{11}, \omega_{12})^*, \quad t \in [-T-T_1, -T].$$

Extending $u(t)$ to each of the two intervals, we have

$$x(t) = e^{\lambda(t-T)}, \quad t \in [T, T+T_1],$$

$$\begin{aligned} y(t) &= \int_T^t d_1 \delta^2 e^{-\lambda(s-T)} e^{-\lambda(t-s)} ds \\ &= d_1 \delta^2 \int_T^t e^{\lambda(T-t)} ds \\ &= d_1 \delta^2 e^{\lambda(T-t)}(t-T), \quad t \in [T, T+T_1], \end{aligned}$$

$$u(T+T_1) = (1, \beta)^* e^{\lambda T_1},$$

$$\beta = d_1 \delta^2 T_1 e^{-2\lambda T_1} = d_1 \delta \delta_1 T_1 e^{-\lambda T_1}$$

and

$$\begin{aligned} y(t) &= \omega_{12} e^{-\lambda(t+T)}, \quad t \in [-T-T_1, -T], \\ x(t) &= \omega_{11} e^{\lambda(t+T)} + \int_{-T}^t c_1 \delta^2 \omega_{12} e^{\lambda(t+T)} ds \\ &= \omega_{11} e^{\lambda(t+T)} + c_1 \delta^2 \omega_{12} e^{\lambda(t+T)}(t+T), \\ &\qquad t \in [-T-T_1, -T], \end{aligned}$$

$$u(-T-T_1) = (\omega_{11} - c_1 \delta^2 \omega_{12} T_1, \omega_{12})^* e^{-\lambda T_1}.$$

Let

$$\begin{aligned} \bar{u}(t) &= e^{-\lambda T_1} u(t) + \beta \frac{\dot{r}(t)}{\lambda \delta_1}, \\ \bar{z}_2(t) &= -(\lambda \delta_1)^{-1} \dot{r}(t), \\ \overline{Z}(t) &= (\bar{u}(t), \bar{z}_2(t)), \quad t \in [-T-T_1, T+T_1]. \end{aligned}$$

By $\delta_1 = \delta e^{-\lambda T_1}$, we have $r(t) = (0, \delta e^{-\lambda(t-T)})^*$ if $t \in [T, T+T_1]$, and $r(t) = (\delta e^{\lambda(t+T)}, 0)^*$ if $t \in [-T-T_1, -T]$. Therefore

$$\overline{Z}(T+T_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\overline{Z}(-T-T_1) = \begin{pmatrix} \varpi_{11} & -1 \\ \varpi_{12} & 0 \end{pmatrix},$$

where $\varpi_{11} = (\omega_{11} - c_1 \delta^2 \omega_{12} T_1) e^{-2\lambda T_1} + \beta$, $\varpi_{12} = \omega_{12}$. It follows directly that

$$\lim_{\delta_1 \to 0} \frac{\varpi_{11}}{\varpi_{12}} = \lim_{T \to +\infty} \frac{\varpi_{11}}{\varpi_{12}} = 0. \qquad \blacksquare$$

Remark 3.1. The parameter ω_{12} can be used as the second criterion for the stability of Γ, that is, the divergence integral B_1 can be replaced by $1 - \omega_{12}$.

Notice that there exists a fundamental solution matrix for the adjoint equation (11):

$$\Phi_1(t) = (Z_1^{-1}(t))^* \triangleq (v(t), \phi_2(t)),$$

where $v(t) = (v_1(t), v_2(t))^*$. Due to $\Phi_1^*(t) Z_1(t) = I$, we know that

$$v(-T) = (0, \omega_{12}^{-1})^*, \quad v(T) = (1, 0)^*.$$

4. Nonlocal Poincaré Map

We now consider the nonlocal Poincaré map induced by the flow of system (1) in a neighborhood of Γ. The technique is similar to the curvilinear coordinate system established near the periodic orbit (see [Chow & Hale, 1982, p. 350]). From the analysis in Sec. 3, we use the moving coordinate frame $\{u(t), z_2(t)\}$ along the homoclinic loop Γ outside a neighborhood of the origin. Similar to the change of variables near the periodic orbit, we use the following coordinate transformation near Γ:

$$Z(t) = r(t) + u(t)n(t), \quad t \in [-T, T], \tag{12}$$

where $u(t) = (u_1(t), u_2(t))^*$. Substituting (12) into (4), and by the notations in (1) [or (2)], we have

$$\dot{r}(t) + u(t)\dot{n}(t) + \dot{u}(t)n(t) = f(r(t) + u(t)n(t)).$$

Using $\dot{r} = f(r(t))$, $\dot{u}(t) = Au(t)$, the above equation is changed into the following form

$$u\dot{n} = f(r + un) - f(r) - Aun.$$

Taking Taylor expansion to the right side of the equation, and multiplying both sides with $v(t) = (v_1, v_2)^*$, by $v(t)u(t) \equiv 1$, we have

$$\begin{aligned}\dot{n} &= A_2(t)n^2 + A_3(t)n^3 + \cdots + A_p(t)n^p \\ &\quad + \cdots + A_l(t)n^l + A_{l+1}(t, \theta, n)n^{l+1},\end{aligned} \tag{13}$$

where

$$\begin{aligned} &p = 2, 3, \ldots, l, \quad l \le \alpha - 3, \quad t \in [-T, T], \\ &A_p(t) = \frac{1}{p!}\left[v_1\left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}\right)^p P \right. \\ &\qquad \left. + v_2\left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}\right)^p Q\right]\Bigg|_{z=r(t)}, \\ &A_{l+1}(t) = \frac{1}{(l+1)!}\left[v_1\left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}\right)^{l+1} P \right. \\ &\qquad \left. + v_2\left(u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y}\right)^{l+1} Q\right]\Bigg|_{z=r(t)+\Delta(t)}, \end{aligned} \tag{14}$$

$\Delta(t) = \theta(t)u(t)n(t)$, $\theta(t) \in (0, 1)$, A_{l+1} is a continuous bounded function of t, θ, and n, while A_p is continuously differentiable with respect to t for $p \le l$.

Notice that when we look for the next loop number, we need to assume the previous loop numbers are zero. Since the first four loop numbers for the stability problem are given, for simplicity, we assume $B_1 = 0$ next. By Theorem 3, that is

$$\omega_{12} = 1. \tag{15}$$

Solving Eq. (13), we obtain the solution $n(t) = n(t, -T, n_0)$ with the initial value n_0 at $t = -T$, then $n(t) = n(t, -T, n_0)$ defines the nonlocal Poincarè map (see Fig. 2)

$$P_1 : S_1^+ \to S_0^+, \quad q_1 \to q_2. \tag{16}$$

According to the definition of S_0 and S_1, Theorem 3, and the coordinate transformation (12), we have

$$\begin{aligned} q_1 &= (\delta, y_1)^* = r(-T) + n_0 u(-T) \\ &= (\delta, 0)^* + n_0(0, 1)^* \\ &= (\delta, n_0)^*, \\ q_2 &= (x_2, \delta)^* = r(T) + n(T)u(T) \\ &= (0, \delta)^* + n(T)(1, 0)^* \\ &= (n(T), \delta)^*. \end{aligned} \tag{17}$$

It is easy to see that the nonlocal map P_1 is uniquely determined by the solution $n(t)$ that satisfies the boundary values $n(-T) = n_0$ and $n(T) = x_2$.

Note that the solution $n(t)$ is at least $l+1$ times continuously differentiable with respect to initial values n_0 on $[0, \varepsilon)$. Hence it can be expanded into a Taylor polynomial with Lagrange remainder

$$\begin{aligned} n(t) &= b_1 n_0 + b_2(t)n_0^2 + b_3(t)n_0^3 + \cdots + b_p(t)n_0^p \\ &\quad + \cdots + b_l(t)n_0^l + b_{l+1}(t, \xi, n_0)n_0^{l+1}. \end{aligned} \tag{18}$$

Here

$$\begin{aligned} &\xi = \xi(t) \in (0, 1), \quad b_1 = 1, \quad b_p(-T) = 0, \\ &b_p(T) = B_p, \quad p = 2, 3, \ldots, l, \quad b_{l+1}(-T, \xi, n_0) = 0 \end{aligned}$$

and $b_{l+1}(t, \xi, n_0) = \frac{\partial^{l+1} n(t, -T, x)}{\partial x^{l+1}}|_{x=\xi(t)n_0}$. Also $b_p(t)$ is a C^1 function in t, and b_{l+1} is a C^1 function in (t, ξ, n_0). Substituting $n(t)$ into (13), we have

$$\begin{aligned} &b_2' n_0^2 + \cdots + b_p' n_0^p + \cdots + b_l'(t)n_0^l \\ &\quad + \frac{\partial}{\partial t} b_{l+1}(t, \xi, n_0)n_0^{l+1} \end{aligned}$$

$$= A_2(b_1 n_0 + b_2(t)n_0^2 + \cdots + b_{l+1}n_0^{l+1})^2 + A_3(n(t))^3 + \cdots + A_p(n(t))^p + \cdots + A_{l+1}(n(t))^{l+1}.$$

Therefore,

$$b_2' = h_2(t) \triangleq A_2(t),$$

$$b_3' = h_3(t) \triangleq A_3 + A_2 C_2^1 b_2,$$

$$b_4' = h_4(t) \triangleq A_4 + A_3 C_3^1 b_2 + A_2(C_2^1 b_3 + C_2^2 b_2^2),$$

$$b_5' = h_5(t) \triangleq A_5 + A_4 C_4^1 b_2 + A_3(C_3^1 b_3 + C_3^2 b_2^2) + A_2(C_2^1 C_1^1 b_2 b_3 + C_2^1 b_4),$$

$$b_6' = h_6(t) \triangleq A_6 + A_5 C_5^1 b_2 + A_4(C_4^1 b_3 + C_4^2 b_2^2) + A_3(C_3^1 C_2^1 b_2 b_3 + C_3^1 b_4 + b_2^3) + A_2(C_2^1 C_1^1 b_2 b_4 + C_2^1 b_5 + b_3^2),$$

$$b_7' = h_7(t) \triangleq A_7 + A_6 C_6^1 b_2 + A_5(C_5^1 b_3 + C_5^2 b_2^2) + A_4(C_4^1 C_3^1 b_2 b_3 + C_4^1 b_4) + A_3(C_3^1 C_2^1 b_2 b_4 + C_3^1 b_5 + C_3^1 b_3^2 + C_3^1 b_2^2 b_3) + A_2(C_2^1 b_6 + C_2^1 C_1^1 b_2 b_5 + C_2^1 C_1^1 b_3 b_4),$$

$$b_8' = h_8(t) \triangleq A_8 + A_7 C_7^1 b_2 + A_6(C_6^1 b_3 + C_6^2 b_2^2) + A_5(C_5^1 b_4 + C_5^1 C_4^1 b_2 b_3) + A_4(C_4^1 b_5 + C_4^1 C_3^1 b_2 b_4 + C_4^2 b_3^2 + C_4^1 C_3^1 b_2^2 b_3) + A_3(C_3^1 b_6 + C_3^1 C_2^1 b_2 b_5 + C_3^1 C_2^1 b_3 b_4 + C_3^1 b_2^2 b_4 + C_3^1 b_3^2) + A_2(C_2^1 b_7 + C_2^1 b_2 b_6 + C_2^1 b_3 b_5 + b_4^2).$$

$$\vdots$$

In general, for $\forall\, p = 2, 3, \ldots, l$, we have the following expression

$$b_p' = h_p(t) \triangleq A_p + A_{p-1} C_{p-1}^1 b_2 + A_{p-2}(C_{p-2}^1 b_3 + C_{p-2}^2 b_2^2) + \cdots + A_q \sum_{i_1+\cdots+i_q=p} b_{i_1} b_{i_2} \cdots b_{i_q} + A_3 \sum_{i_1+i_2+i_3=p} b_{i_1} b_{i_2} b_{i_3} + A_2 \sum_{i_1+i_2=p} b_{i_1} b_{i_2}, \tag{19}$$

where $q = 4, 5, \ldots, p-3$, the symbol $\sum_{i_1+\cdots+i_q=p} \times b_{i_1} b_{i_2} \cdots b_{i_q}$ represents the sum of all the permutations of the function column $b_{i_1} b_{i_2} \cdots b_{i_q}$ that satisfies the condition $i_1 + \cdots + i_q = p$.

For example, when $p = 5$, the coefficient of A_3 is

$$\sum_{i_1+i_2+i_3=p} b_{i_1} b_{i_2} b_{i_3} = b_1 b_1 b_3 + b_1 b_3 b_1 + b_3 b_1 b_1 + b_1 b_2 b_2 + b_2 b_2 b_1 + b_2 b_1 b_2 = 3b_3 + 3b_2^2 = C_3^1 b_3 + C_3^2 b_2^2.$$

When $p = 6$, the coefficient of A_4 is

$$\sum_{i_1+i_2+i_3+i_4=p} b_{i_1} b_{i_2} b_{i_3} b_{i_4} = b_1 b_1 b_1 b_3 + b_1 b_1 b_3 b_1 + b_1 b_3 b_1 b_1 + b_3 b_1 b_1 b_1 + b_1 b_1 b_2 b_2 + b_1 b_2 b_1 b_2 + b_1 b_2 b_2 b_1 + b_2 b_1 b_1 b_2 + b_2 b_1 b_2 b_1 + b_2 b_2 b_1 b_1 = 4b_3 + 6b_2^2 = C_4^1 b_3 + C_4^2 b_2^2.$$

Using the expression (19), the coefficients $b_2(t), b_3(t), \ldots, b_l(t)$ in the Taylor expansion (18) can be determined one by one, with the initial values $b_p(-T) = 0$. In order to determine the stability of Γ, we need to solve $B_p \triangleq b_p(T)$, $p = 2, \ldots, l$. Hence, By (18), (19) we obtain

Theorem 4.

$$B_2 = \int_{-T}^{T} A_2(t)dt,$$

$$B_3 = \int_{-T}^{T} A_3(t)dt + B_2^2,$$

$$B_4 = \int_{-T}^{T} (A_4 + A_3 b_2)dt + 2B_2 B_3 - B_2^3,$$

$$\vdots$$

Proof. B_1 and B_2 can be obtained directly from the definition. By $A_2 = b_2'$, $b_2(-T) = 0$, we have

$$B_3 = b_3(T) = \int_{-T}^{T} A_3(t)dt + 2\int_{-T}^{T} A_2 b_2 dt$$